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Nuclear and String Fields on a Kähler Manifold

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An arbitrary oriented Riemannian manifold of real dimension two is a complex manifold that is also the world sheet of an oriented closed string. Another example is the complex Grassmann manifold of p-planes in C^{2p}, which is shown to carry the most symmetric state of ^{9}Li. In both cases we are concerned with a chiral spinor field on a curved surface that gives rise to anyons in the nuclear case. Specifically we find a distorted torus on a Kähler manifold which is also a Calabi–Yau space.

1. INTRODUCTION

This paper emphasizes the common geometry of string fields and nuclear surfaces, that is, two-dimensional surfaces upon which nucleons move in certain states. An example is given by ^{9}Li analyzed in Section 3. It is well known that particles with spin and parity moving on a surface will exhibit fractional statistics (Lerda, 1992; Wen *et al.*, 1989). Therefore it should come as no surprise that fractional statistics have been calculated for ^{9}Li. The analysis is reviewed in Section 3. However, spinor string fields are also supposed to admit positive and negative chirality (Green *et al.*, 1988), so perhaps it should be expected that they require the same geometry, namely a two-dimensional surface on a Kähler manifold. In this way geometry replaces mechanics and it is revealing that the same manifold should emerge from two completely different approaches.

The methods used to analyze the nuclear case have been reviewed in Section 1 of de Wet (1994), so only enough background to show the connection with Kähler spaces will be given here.

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We construct the tensor products, in the enveloping algebra $A(\gamma)$ of the Dirac ring, of an irreducible self-representation

$$\frac{1}{4}\Psi = (iE_4\psi_1 + E_{23}\psi_2 + E_{14}\psi_3 + E_{05}\psi_4)e \tag{1.1}$$

with itself. Here Eddington's E-numbers are related to the Dirac matrices by

$$\gamma_\nu = iE_{0\nu}, \quad E_{\mu\nu} = E_{\sigma\mu}E_{\sigma\nu}, \quad E_{\mu\nu}^2 = -1, \quad E_{\mu\nu} = -E_{\nu\mu},$$
$$\mu < \nu = 1, \ldots, 5$$

and the commuting operators E_{23}, E_{14}, E_{05} are, respectively, independent infinitesimal rotations in 3-space, 4-space, and isospace that correspond to the spin σ, parity π, and charge T_3 carried by a single nucleon. The parameters ψ_2, ψ_3, ψ_4 are half-angles of rotation and e is a primitive idempotent of the Dirac ring; E_4 is the unit matrix.

A rotation through 180° about x will change spin up to spin down and if this is followed by a rotation of 180° about t, x can go to $-x$ without inverting time, but instead changing to a left-handed coordinate system, in other words, causing a parity reversal ($E_{14} \to -E_{14}$).

The basis elements of $A(\gamma)$ are the $4^A \times 4^A$ matrices ($A = N + Z$)

$$E_{\mu\nu}^l = E_4 \otimes \cdots \otimes E_4 \otimes E_{\mu\nu} \otimes E_4 \otimes \cdots \otimes E_4$$

with $E_{\mu\nu}$ in the lth position. The elements $E_{\mu\nu}^l$ and $E_{\mu\nu}^{l+1}$ commute and $A(\gamma)$ is found to have the following generators:

$$\Gamma_\nu^{(A)} = \frac{1}{2}(E_{0\nu}^1 + E_{0\nu}^2 + \cdots + E_{0\nu}^4), \qquad \nu = 1, \ldots, 5 \tag{1.2a}$$

$$\sigma_{\mu\nu}^{(A)} = [\Gamma_\mu^{(A)}, \Gamma_\nu^{(A)}] = \frac{1}{2}(E_{\mu\nu}^1 + E_{\mu\nu}^2 + \cdots + E_{\mu\nu}^4) \tag{1.2b}$$

$$\eta_\nu^{(A)} = E_{0\nu} \otimes \cdots \otimes E_{0\nu} = E_{0\nu}^1 E_{0\nu}^2 \cdots E_{0\nu}^A \tag{1.2c}$$

$$\eta_{\mu\nu}^{(A)} = \eta_\mu^{(A)}\eta_\nu^{(A)} = E_{\mu\nu}^1 E_{\mu\nu}^2 \cdots E_{\mu\nu}^A, \qquad \mu < \nu = 1, \ldots, 5 \tag{1.2d}$$

Then the irreducible representations or minimal left ideals of $A(\gamma)$ are

$$\Psi^{(A)} = \sum_\lambda C_{[\lambda]} P_{[\lambda]} \tag{1.3}$$

with

$$C_{[\lambda]} = i^{\lambda_1} C(E_{23}^1 \cdots E_{23}^{\lambda_2} E_{14}^{\lambda_2+1} \cdots E_{14}^{\lambda_2+\lambda_3} E_{05}^{\lambda_2+\lambda_3+1} \cdots E_{05}^{A-\lambda_1}) \tag{1.4}$$

if C denotes summation over the

$$N_{[\lambda]} = A!/(\lambda_1!\ \lambda_2!\ \lambda_3!\ \lambda_4!)$$

combinations of the basis elements contained in the bracket. Here $[\lambda]$ is a partition

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = A \tag{1.5}$$

and

$$\begin{aligned} P_{[\lambda]} = i^{-A}(i^A \psi_1^{\lambda_1}\psi_2^{\lambda_2}\psi_3^{\lambda_3}\psi_4^{\lambda_4} + \eta_{23}^{(A)}\psi_2^{\lambda_1}\psi_1^{\lambda_2}\psi_4^{\lambda_3}\psi_3^{\lambda_4} \\ + \eta_{14}^{(A)}\psi_3^{\lambda_1}\psi_4^{\lambda_2}\psi_1^{\lambda_3}\psi_2^{\lambda_4} + \eta_5^{(A)}\psi_4^{\lambda_1}\psi_3^{\lambda_2}\psi_2^{\lambda_3}\psi_1^{\lambda_4})\epsilon_A \end{aligned} \tag{1.6a}$$

is a projection operator satisfying

$$P_{[\lambda]}^2 = P_{[\lambda]}\psi, \qquad \psi \equiv \psi_1\psi_2\psi_3\psi_4 \tag{1.6b}$$

Also $\epsilon_A = e \otimes \cdots \otimes e = e^1 e^2 \cdots e^A$ is a primitive idempotent in $A(\gamma)$, so that equation (1.6a) has the same form for A nucleons as the basic relation (1.1).

Each partition (1.5) represents a charge-spin state of the nucleus. By choosing 4×4 matrix representations for E_{23}, E_{14}, E_{05} and factoring out those configurations that are the same up to a rearrangement that does not affect the net spin, parity, or charge, it may be shown (de Wet, 1973) that $C_{[\lambda]}$ decomposes beautifully into subspaces constituting isobaric multiplets characterized by the neutron and proton numbers $N = (\lambda_1 + \lambda_2)$, $Z = (\lambda_3 + \lambda_4)$. If we focus on one member of the multiplet, the first two terms of (1.6a), with the same values of Z, N, will belong to the same nucleus; but in the third and fourth terms $(\lambda_3 + \lambda_4)$ has replaced $(\lambda_1 + \lambda_2)$, so charge has been reversed and a mirror nucleus emerges.

The possible states $[\lambda] = [\lambda_1\lambda_2\lambda_3\lambda_4]$ of the mirror nuclei ^{9}Li and ^{9}C are set out in Table I, which follows the scheme of (1.6a). We note that the first and second, third and fourth terms differ in that $(\lambda_2 + \lambda_3)$ replaces $(\lambda_1 + \lambda_4)$. Thus if $(\lambda_2 + \lambda_3)$ is the number of particles with a given spin, there will be two possible spin states s for each nucleus. Finally, if $(\lambda_2 + \lambda_4)$ is the number of particles with a given parity p, these changes lead automatically to parity reversals. In Table I the values of

$$2\sigma_{23}^{(A)} = \sigma_0 = 2is, \qquad \pi_0 = 2ip = 2\sigma_{14}^{(A)}$$

have been obtained by assuming that $(\lambda_2 + \lambda_3)$ is the number of nucleons with a negative spin and $(\lambda_2 + \lambda_4)$ the number with positive parity. It is then possible to substitute directly into equation (3.5) to find the eigenvalues of $C_{[\Lambda]}$ and compare them with a computer calculation based on the matrix representation (3.6) of σ_0, π_0. The fact that the agreement is exact confirms the canonical labeling adopted and enables each row of the matrix $C_{[\Lambda]}$ to be labeled.

The $C_{[\lambda]}$ of (1.4) are invariant operators, but it will be shown in Section 3 that it is possible to choose one operator $C_{[\Lambda]}$ (belonging to the fundamental state $[\Lambda] = [\Lambda_1\Lambda_2\Lambda_3\Lambda_4]$ which need not be the ground state) that is also *CP*-invariant. Special cases of $C_{[\lambda]}$ have already been encountered in (1.2), namely

$$\begin{aligned} C_{[(A-1)100]} &= 2i^{(A-1)}\sigma_1 \\ C_{[(A-1)010]} &= 2i^{(A-1)}\pi_1 \\ C_{[(A-1)001]} &= 2i^{(A-1)}\Gamma_3^{(A)} \end{aligned} \tag{1.7}$$

where

$$i\sigma_1 \equiv i\sigma_{23}^{(A)} = s_1, \qquad i\pi_1 \equiv i\sigma_{14}^{(A)} = p$$

The matrix representations

$$\begin{aligned} \sigma_{jk}^{(A)} &\equiv \sigma_i = E_N \otimes {}^P\Gamma_i + {}^N\Gamma_i \otimes E_P \\ \sigma_{i4}^{(A)} &\equiv \pi_i = E_N \otimes {}^P\Gamma_i - {}^N\Gamma_i \otimes E_P, \qquad i = 1, 2, 3 \end{aligned} \tag{1.8}$$

are just those of Biedenharn and Louck (1981) for a coupled system of spinning protons and neutrons and provide the required connection between equation (1.1) and nuclear theory. Here ${}^P\Gamma_i$ and ${}^N\Gamma_i$ are $(P + 1)$- and $(N + 1)$-dimensional Lie operators of SO_3 and E_P and E_N are $(P + 1)$ and $(N + 1)$ unit matrices. Also, σ_i and π_i are the infinitesimal operators of the four-dimensional rotation group O_4, so we have found a supermultiplet augmented by the invariant operators $C_{[\Lambda]}$.

This paper is about nuclear trajectories on a curved surface, but apart from this motion, it will emerge in Section 3 that the entire distorted torus of Fig. 1 also rotates about the axis $X_4 = -X_5$, thereby imparting orbital angular momentum to the system.

The operators $\exp(C_{[\lambda]})$ are representations of the orthogonal group $O(p)$ if $(\lambda_2 + \lambda_3)$ is even, but representations of $SO(p + q)$ if $(\lambda_2 + \lambda_3)$ is odd. Here p, q are greater than 2 and

$$\begin{aligned} p = q &= \frac{1}{2}(Z + 1)(N + 1) \\ &\qquad \text{if} \quad Z \text{ or } N\text{, or both, are odd} \\ &= (q + 1) = \frac{1}{2}\{(Z + 1)(N + 1) + 1\} \\ &\qquad \text{if} \quad \text{both } Z, N \text{ are even} \end{aligned} \tag{1.9}$$

The matrices $SO(p + q)$ rotate the p-planes in a complex Grassmann manifold (Kobayashi and Nomizu, 1969, Chapter IX, equation 6.4) and in the following sections the theorem on exponentiation of a finite matrix developed by de Wet (1994) will be used to analyze the differential geometry of this manifold (also called a Kähler manifold). In particular an expression will be derived in Section 3 for the connection coefficients from which it follows that the components of the Ricci curvature tensor vanish. Using a corollary due to Kobayashi and Nomizu (1969, Chapter X, §2), we can find a two-dimensional surface on the Kähler manifold that carries the nucleons of ^{9}Li.

2. MEASURE ON A COMPLEX GRASSMANN MANIFOLD

In this section we will find a corollary to the theorem derived by de Wet (1994) on the exponentiation of Lie algebras. This theorem is valid for all representations of the Lie algebra of a compact Lie group G and in the case of a noncompact group may be adapted by using the dual representation $\theta \to i\theta$ so that the circular functions $\sin\theta$, $\cos\theta$ map into their hyperbolic counterparts $\sinh\theta$, $\cosh\theta$. However, here we are concerned with orthogonal groups and will find a simple orthogonality relationship that enables the coefficients a_i, b_i of the exponential expansion to be expressed in closed form. We can then go on to confirm a metric proposed by Wong for the complex Grassmann manifold.

Theorem.

I. If μ is a class 1 matrix with the characteristic equation

$$\mu(\mu - 1)(\mu - \lambda_2) \cdots (\mu - \lambda_n) = 0 \tag{2.1a}$$

then we may write

$$e^{i\mu\theta} = 1 + \sum_{j=1,2,\ldots}^{n} \mu^j \Sigma_j(\theta) + i \sum_{j=1,2,\ldots}^{n} \mu^j S_j(\theta) \tag{2.1b}$$

where

$$\Sigma_j(\theta) = b_{j0} + b_{j1}\cos\theta + b_{j2}\cos\lambda_2\theta + \cdots + b_{jn}\cos\lambda_n\theta \tag{2.1c}$$

$$S_j(\theta) = b_{j1}\sin\theta + b_{j2}\sin\lambda_2\theta + \cdots + b_{jn}\sin\lambda_n\theta \tag{2.1d}$$

II. If μ is a class 2 matrix with the characteristic equation

$$\mu(\mu^2 + 1)(\mu^2 + \lambda_2^2) \cdots (\mu^2 + \lambda_n^2) = 0 \tag{2.2a}$$

then

$$e^{\mu\theta} = 1 + \sum_{j=2,4,\ldots}^{2n} \mu^j \Sigma_j(\theta) + \sum_{j=1,3,\ldots}^{2n-1} \mu^j S_j(\theta) \tag{2.2b}$$

where

$$i^{j-1} S_j(\theta) = a_{j1}\sin\theta + a_{j2}\sin\lambda_2\theta + \cdots + a_{jn}\sin\lambda_n\theta \tag{2.2c}$$

and

$$\Sigma_{j+1}(\theta) = \int S_j(\theta)\, d\theta \tag{2.2d}$$

The eigenvalues $\lambda_2, \ldots, \lambda_n$ are positive, and it always is possible to reduce any set of eigenvalues to the positive canonical form 0, 1, $\lambda_2, \ldots, \lambda_n$ by subtracting a constant λ_t to translate the spectrum and then dividing by a factor λ_f so that one eigenvalue of the translated spectrum has the value unity. This follows because if $AX = \lambda X$, then $(A - \lambda_t)X = (\lambda - \lambda_t)X$.

An elementary application of equations (2.1) and (2.2) is to find representations of SO_3. In the case of $j = 2$ the characteristic equation is

$$\mu(\mu^2 + 1)(\mu^2 + 4) = 0$$

and the representation $D^{(2)}_{m'm}(\pi/2, \theta, \pi)$ is given by

$$\begin{aligned} e^{\mu\theta} = {} & \frac{1}{4}(\mu^2 + 1)(\mu^2 + 4) - \frac{\mu^2}{3}(\mu^2 + 4)\cos\theta + \frac{\mu^2}{12}(\mu^2 + 1)\cos 2\theta \\ & + \frac{\mu}{3}(\mu^2 + 4)\sin\theta - \frac{\mu}{6}(\mu^2 + 1)\sin 2\theta \end{aligned} \tag{2.3}$$

with

$$\begin{aligned} \langle jm + 1|\mu|jm\rangle &= [(j - m)(j + m + 1)]^{1/2} \\ \langle jm - 1|\mu|jm\rangle &= -[(j + m)(j - m + 1)]^{1/2} \end{aligned} \tag{2.3a}$$

However, we can also write (2.3) as

$$\begin{aligned} e^{\mu\theta} = {} & \frac{F_0(\mu)}{F_0(0)} + \frac{\mu F_1(\mu)}{iF_1(i)}\cos\theta + \frac{\mu F_2(\mu)}{2iF_2(2i)}\cos 2\theta \\ & + \frac{iF_1(\mu)}{F_1(i)}\sin\theta + \frac{iF_2(\mu)}{F_2(2i)}\sin 2\theta \end{aligned} \tag{2.4}$$

if

$$\begin{aligned} F(\mu) &\equiv \mu(\mu^2 + 1)(\mu^2 + 4) = 0 \\ F_0(\mu) &\equiv F(\mu)/\mu \\ F_1(\mu) &\equiv F(\mu)/(\mu^2 + 1) \\ F_2(\mu) &\equiv F(\mu)/(\mu^2 + 4) \end{aligned} \tag{2.5}$$

The $F_j(\mu)$ are orthogonal functions satisfying

$$F_j(\mu)F_k(\mu) = 0 \tag{2.6}$$

and the following is a generalization of (2.4).

Corollary. In the general case where

$$F(\mu) = \mu(\mu^2 + 1)(\mu^2 + \lambda_2^2) \cdots (\mu^2 + \lambda_n^2) = 0$$

$$F_k(\mu) = F(\mu)/(\mu^2 + \lambda_k^2)$$

$$F_0(\mu) = F(\mu)/\mu$$

we have

$$e^{\mu\theta} = \frac{F_0(\mu)}{F_0(0)} + \mu \sum_{k=1,2,\ldots}^{n} \frac{F_k(\mu)}{i\lambda_k F(i\lambda_k)} \cos \lambda_k\theta + i \sum_{k=1,2,\ldots}^{n} \frac{F_k(\mu)}{F_k(i\lambda_k)} \sin \lambda_k\theta \tag{2.7}$$

Proof. In the diagonal representation

$$\mu_d = \mathrm{diag}(0, i, i\lambda_2, \ldots, i\lambda_n)$$

(2.7) becomes

$$e^{\mu_d\theta} = \sum_{k=0,1,2,\ldots}^{n} \frac{F_k(\mu_d)}{F(i\lambda_k)} e^{i\lambda_k\theta} = \mathrm{diag}(1, e^{i\theta}, \ldots, e^{i\lambda_n\theta}) \tag{2.7a}$$

because $F_k(\mu_d)/F_k(i\lambda_k)$ will assign a one to each row of the identity matrix E. Here repeated roots are treated as though they are independent, so if λ_k is n-fold degenerate, there will be n terms $e^{i\lambda^k\theta}$ on the right-hand side of (2.7a). This confirms (2.7) and because $F_j(\mu)F_k(\mu) = 0$, we have

$$e^{\mu_d\theta}[e^{\mu_d\theta}]^\dagger = \sum_{k=0,1,\ldots}^{n} \left(\frac{F_k(\mu_d)}{F_k(i\lambda_k)}\right)^2 = E \tag{2.8}$$

In the general case there is some orthogonal transformation S such that

$$\mu = S\mu_d S^{-1}, \qquad e^{\mu\theta} = Se^{\mu_d\theta}S^{-1}$$

but

$$\begin{aligned} SF_k(\mu_d)S^{-1} &= S\mu_d S^{-1}S(\mu_d^2 + 1)S^{-1}S(\mu_d^2 + \lambda_2^2)S^{-1} \cdots S(\mu_d^2 + \lambda_n^2)S^{-1} \\ &= \mu(\mu^2 + 1)(\mu^2 + \lambda_2^2) \cdots (\mu^2 + \lambda_n^2) = F_k(\mu) \end{aligned}$$

[where the factor $(\mu^2 + \lambda_k^2)$ is omitted]. Therefore writing (2.7a) as

$$e^{\mu_d\theta} = \frac{F_0(\mu_d)}{F_0(0)} + \mu_d \sum_{k=1,2,\ldots}^{n} \frac{F_k(\mu_d)}{i\lambda_k F_k(i\lambda_k)} \cos \lambda_k\theta + i \sum_{k=1,2,\ldots} \frac{F_k(\mu_d)}{F_k(i\lambda_k)} \sin \lambda_k\theta$$

and operating on both sides with S, S^{-1}, we again obtain (2.7). This completes the proof, and it may similarly be shown that in the case of the class 1 matrices where

$$F_k(\mu) = \mu(\mu - 1)(\mu - \lambda_2) \cdots (\mu - \lambda_n)/(\mu - \lambda_k)$$

we have

$$e^{i\mu\theta} = \sum_{k=0,1,\ldots}^{n} \frac{F_k(\mu)}{F_k(\lambda_k)} e^{i\lambda_k\theta}, \qquad F_j(\mu)F_k(\mu) = 0 \tag{2.9}$$

$$e^{i\mu\theta}[e^{i\mu\theta}]^\dagger = \sum_{k=0,1,\ldots}^{n} \left[\frac{F_k(\mu)}{F_k(\lambda_k)}\right]^2 = E \tag{2.9a}$$

Equations (2.7) and (2.9) are a restatement of the nuclear spectral theorem for pure states, because by virtue of the state labeling summarized in the introduction, each state $[\lambda_1\lambda_2\lambda_3\lambda_4]$ yields an eigenvalue λ_k of $C_{[\lambda]}$ determined precisely by the numbers of nucleons with given spin, parity, and charge. Thus the rows of the matrix μ, which are the states $[\lambda]$, may be labeled by $z_k = i\lambda_k\theta$.

We can now use (2.7) to find a measure g on the complex Grassmann manifold by following Kobayashi and Nomizu (1969, Chapter IX, §6) and Wong (1967). On these manifolds μ has the complex structure

$$\begin{bmatrix} & -A \\ A & \end{bmatrix}$$

where A is a real $p \times p$ matrix, and (2.7) shows that it is possible to write

$$e^{\mu\theta} = Z_0(\cos\theta) + Z_1(\sin\theta) = \begin{bmatrix} Z_0 & -Z_1 \\ Z_1 & Z_0 \end{bmatrix} \tag{2.9b}$$

because Z_0 depends only on the even powers of μ, and Z_1 only on the odd. Then if $T = Z_1 Z_0^{-1}$,

$$ds^2 = \mathrm{Tr}\, \frac{dT}{(1 + T\overline{T}^t)} \frac{\overline{dT}^t}{(1 + T\overline{T}^t)} \tag{2.10}$$

where $\overline{T}^t$, $\overline{dT}^t$ are the conjugate transposes of T, dT (ibid.). At $T = 0$ this reduces to the metric

$$ds^2 = \mathrm{Tr}\, dT\, \overline{dT}^t \tag{2.10a}$$

To evaluate (2.10), we need some special properties of the orthogonal functions $F(\mu)$. To begin with, set $\theta = 0$ in (2.7) and square to find

$$1 = \frac{F_0(\mu)}{F_0(0)} + \mu^2 \sum_{k=1}^{n} \frac{(F_k(\mu)/\mu)}{i\lambda_k F(i\lambda_k)}$$
$$= \left[\frac{F_0(\mu)}{F_0(0)}\right]^2 + \mu^4 \sum_{k=1}^{n} \left[\frac{(F_k(\mu)/\mu)}{i\lambda_k F_k(i\lambda_k)}\right]^2$$

Comparing coefficients, we find

$$[\mu(F_k(\mu)/\mu)]^2 = i\lambda_k F_k(i\lambda_k)(F_k(\mu)/\mu) \tag{2.11a}$$

Now because

$$F(\mu) = \mu(\mu^2 + 1)(\mu^2 + \lambda_2^2) \cdots (\mu^2 + \lambda_k^2) \cdots (\mu^2 + \lambda_n^2) = 0$$

we find that

$$\mu^3(\mu^2 + 1)(\mu^2 + \lambda_2^2) \cdots (\mu^2 + \lambda_n^2) = -\lambda_k^2 \mu(\mu^2 + 1)(\mu^2 + \lambda_2^2) \cdots (\mu^2 + \lambda_n^2)$$

or

$$\mu^2(F_k(\mu)/\mu) = -\lambda_k^2(F_k(\mu)/\mu) = \frac{i\lambda_k[\mu(F_k/\mu)]^2}{F_k(i\lambda_k)}$$

by (2.11a). Then

$$\left[\frac{F_k(\mu)}{\mu}\right]^2 = \frac{F_k(i\lambda_k)}{(i\lambda_k)} \frac{F_k(\mu)}{\mu} \tag{2.11b}$$

and

$$K_k(\mu) \equiv \frac{(i\lambda_k)}{F_k(i\lambda_k)} \frac{F_k(\mu)}{\mu} \tag{2.11c}$$

is idempotent,

$$K_k^2(\mu) = K_k(\mu), \qquad K_k(\mu)K_j(\mu) = 0 \tag{2.11d}$$

Only when $\mu = \mu_d$ do we find

$$K_k(\mu) = 1, \qquad K \equiv \sum_k K_k(\mu) = E \tag{2.11e}$$

Returning to (2.10),

$$Z_0^{-1} = \frac{F_0(\mu)}{F_0(0)} + \mu \sum_{k=1,2,\ldots}^{p} \frac{F_k(\mu)}{i\lambda_k F(i\lambda_k)} (\cos \lambda_k\theta)^{-1}$$

and the odd powers of μ yield

$$Z_1 = i \sum_{k=1,\ldots} \frac{F_k(\mu)}{F_k(i\lambda_k)} \sin \lambda_k\theta$$

Thus

$$T = Z_1 Z_0^{-1} = -\bar{T}^t = \mu \sum_{k=1,2,\dots}^{n} \frac{i(F_k(\mu)/\mu)}{F_k(i\lambda_k)} \tan \lambda_k\theta \tag{2.12}$$

by (2.11a), and

$$T\bar{T}^t = \sum_{k=1,\dots}^{n} K_k(\mu) \tan^2\lambda_k\theta \tag{2.12a}$$

again by (2.11a). Now apply the idempotent property (2.11d) to write

$$\begin{aligned}
(1 + x)^{-1} &= 1 - x + x^2 - x^3 + \cdots; \qquad x \equiv T\bar{T}^t \\
&= 1 + K_1(\mu)(-\tan^2\theta + \tan^4\theta - \tan^6\theta + \cdots) \\
&\quad + K_2(\mu)(-\tan^2\lambda_2\theta + \tan^4\lambda_2\theta - \tan^6\lambda_2\theta + \cdots) \\
&\quad + \cdots \\
&\quad + K_n(\mu)(-\tan^2\lambda_n\theta + \tan^4\lambda_n\theta - \tan^6\lambda_n\theta + \cdots) \\
&= 1 + K_1(\mu)[(1 + \tan^2\theta)^{-1} - 1] + K_2(\mu)[(1 + \tan^2\lambda_2\theta)^{-1} - 1] \\
&\quad + \cdots + K_n(\mu)[(1 + \tan^2\lambda_n\theta) - 1]
\end{aligned} \tag{2.12b}$$

By (2.12)

$$dT = \mu \, d\theta \sum_k K_k(\mu) \sec^2\lambda_k\theta$$

so that

$$dT/(1 + T\bar{T}^t) = \mu \, d\theta \, K = \mu_d \, d\theta \tag{2.13}$$

and (2.10) yields

$$ds^2 = \operatorname{Tr} \mu_d\bar{\mu}_d \, d\theta^2 = \sum_{k=1,2}^{p} dz_k \, \overline{dz_k} \tag{2.13a}$$

for the diagonal case. This is simply the differential of the sum of the squares of the n angles $\lambda_k\theta$ between two consecutive p planes in the $2p$-dimensional Euclidean space E^{2p} in accord with a theorem due to Wong (1967). Since there is now more than one angle, we find more than one angular momentum series, as will be illustrated by ^{9}Li in Section 3. These fractional chiral-spin statistics square with the fact that the nucleons are moving on a surface (Lerda, 1992).

Equation (2.13a) is a flat measure which is known to be carried by the torus. In fact the closure of geodesics in a submanifold of the Kähler manifold has been shown by Wong (1967, Theorem 14) to be analytically homeomorphic with a torus if one or more of the eigenvalues λ_k are zero. In the next section ^{9}Li will serve as an example where it is necessary to introduce a translation λ_t which distorts the flat metric.

3. THE MANIFOLD OF LITHIUM-9

In this section we will look at the geodesics obtained by de Wet (1994) for ^{9}Li in a different light. Instead of considering these to be the paths of mesons, we will take the more logical view that the nucleons themselves move on a two-dimensional geodesic surface and can therefore behave as anyons. This is a surface on a complex Grassmann or Kähler manifold which is also known to be the world sheet of an oriented closed string (Green *et al.*, 1988, Chapter 15). Thus there is an intimate relationship between string fields and the motion of nucleons (in certain states) which reinforces the view that strings are supposed to be the ultimate constituents of matter.

The CP-invariant operator for ^{9}Li is $C_{[3303]}$, where $[\Lambda_1\Lambda_2\Lambda_3\Lambda_4] = [3303] \equiv [\Lambda]$ is also the ground state with spin 3/2. We rewrite (1.4) in the form

$$C_{[\Lambda]} = i^{\Lambda_1}\sigma_0^{\Lambda_2}\pi_0^{\Lambda_3}T_0^{\Lambda_4} - \sum_{\lambda} i^{\lambda_1}\sigma_0^{\lambda_2}\pi_0^{\lambda_3}T_0^{\lambda_4} \tag{3.1}$$

where

$$\begin{aligned}\sigma_0 &\equiv 2\sigma_{23}^{(A)} = (E_{23}^1 + \cdots + E_{23}^A)\\ \pi_0 &\equiv 2\sigma_{14}^{(A)} = (E_{14}^1 + \cdots + E_{14}^A)\\ T_0 &\equiv 2\Gamma_5^{(A)} = (E_{05}^1 + \cdots + E_{05}^A)\end{aligned} \tag{3.2a}$$

are related to the real quantum $s, p, T_3 = \frac{1}{2}(Z - N)$ of spin, parity, and charge by

$$\sigma_0 = 2is, \qquad \pi_0 = 2ip, \qquad T_0 = 2iT_3 \tag{3.2b}$$

The summation contains all those terms arising from repeated indices $E_{23}^jE_{23}^j$, $E_{23}^jE_{14}^j$, $E_{23}^jE_{05}^j$, $E_{14}^jE_{05}^j$ that yield a single term by the multiplication table

$$\begin{array}{c|ccc} & E_{23}^j & E_{14}^j & E_{05}^j \\ \hline E_{23}^j & i^2 & iE_{05}^j & iE_{14}^j \\ E_{14}^j & iE_{05}^j & i^2 & iE_{23}^j \\ E_{05}^j & iE_{14}^j & iE_{23}^j & i^2 \end{array} \tag{3.3}$$

Ultimately (3.1) will be expressed in the bilinear form $\Lambda(\sigma_0, \pi_0)$ because T_0 is just the diagonal matrix $i(Z - N)$. An elementary example is

$$\sigma_0T_0 = P(E_{23}^iE_{05}^j) + i\pi_0 \tag{3.4a}$$

Table I. Coherent States of ^{9}Li, ^{9}C

^{9}Li		^{9}C							
p −	+	−	+						
s +	−	−	+	^{9}Li		^{9}C		$C_{[3303]}$	
$\lambda_1\lambda_2\lambda_3\lambda_4$	$\lambda_2\lambda_1\lambda_4\lambda_3$	$\lambda_3\lambda_4\lambda_1\lambda_2$	$\lambda_4\lambda_3\lambda_2\lambda_1$	σ_0	π_0	σ_0	π_0	$= C_{[3033]}$	$C_{[3033]}/16$
6003*	0630	0360	3006	$9i$	$-3i$	$9i$	$3i$	$160i$	$10i$
6012	0621	1260	2106	$7i$	$-5i$	$7i$	$5i$	$80i$	$5i$
6021*	0612	2160	1206	$5i$	$-7i$	$5i$	$7i$	$-80i$	$-5i$
6030	0603	3060	0306	$3i$	$-9i$	$3i$	$9i$	$-160i$	$-10i$
5103*	1530	0351	3015	$7i$	$-i$	$7i$	i	$40i$	$2.5i$
5112	1521	1251	2115	$5i$	$-3i$	$5i$	$3i$	$40i$	$2.5i$
5121*	1512	2151	1215	$3i$	$-5i$	$3i$	$5i$	$-40i$	$-2.5i$
5130	1503	3051	0315	i	$-7i$	i	$7i$	$-40i$	$-2.5i$
4203*	2430	0342	3024	$5i$	i	$5i$	$-i$	$-32i$	$-2i$
4212	2421	1242	2124	$3i$	$-i$	$3i$	i	$16i$	i
4221*	2412	2142	1224	i	$-3i$	i	$3i$	$-16i$	$-i$
4230	2403	3042	0324	$-i$	$-5i$	$-i$	$5i$	$32i$	$2i$
Λ 3303*	3330	0333 Λ	3033	$3i$	$3i$	$3i$	$-3i$	$-56i$	$-3.5i$
3321*	3312	2133	1233	$-i$	$-i$	$-i$	$+i$	$-8i$	$-\frac{1}{2}i$

where P denotes summation over the $A!/(A-n)!$ permutations of the n bracketed generators. Then

$$C_{[(A-2)101]}/i^{(A-2)} = P(E^i_{23}E^j_{05}) = \sigma_0 T_0 - i\pi_0 \tag{3.4b}$$

is an extension of the basic relation (1.7).

In the case ^{9}Li we "add" another nucleon by multiplying (3.4a) by $\sigma_0 = (E^1_{23} + \cdots + E^9_{23})$ to obtain

$$\sigma_0^2 T_0 = P(E^i_{23}E^j_{23}E^k_{05}) + 2iP(E^i_{23}E^j_{14}) + 9i^2 T_0$$

and continue the process until ultimately the invariant operator for ^{9}Li is

$$\begin{aligned} C_{[3303]} &= i^3 P(E^i_{23}E^j_{23}E^k_{23}E^l_{05}E^m_{05}E^n_{05})/(3!\ 3!) \\ &= \frac{1}{6}(\sigma_0^3 + \pi_0^3) + \frac{3}{2}(\sigma_0\pi_0^2 + \sigma_0^2\pi_0) + \frac{17}{3}(\sigma_0 + \pi_0) \end{aligned} \tag{3.5a}$$

after writing $T_0 = i(Z - N) = -3i$. In the case of the mirror nucleus ^{9}C, $T_0 = 3i$ and

$$C_{[3033]} = \frac{1}{6}(\sigma_0^3 - \pi_0^3) + \frac{3}{2}(\sigma_0\,\pi_0^2 - \sigma_0^2\pi_0) + \frac{17}{3}(\sigma_0 - \pi_0) \tag{3.5b}$$

These operators are manifestly CP-symmetric because $T_3 \to -T_3$ is accompanied by $\pi_0 \to -\pi_0$ and moreover their matrix representations are identical up to a rearrangement of rows and columns.

Table I is an evaluation of $C_{[3303]}$ and $C_{[3033]}$ for the coherent states of ^{9}Li and ^{9}C, where we have assumed that $(\lambda_2 + \lambda_3)$ is the number of states with a negative spin and $(\lambda_2 + \lambda_4)$ the number with a positive parity. The first and fourth columns $[\lambda_1\lambda_2\lambda_3\lambda_4]$ and $[\lambda_4\lambda_3\lambda_2\lambda_1]$ have been used to find σ_0 and π_0. If the second and third columns had been chosen, both spins σ_0 and parities π_0 would have changed sign, which would have simply reversed the sign of $C_{[\Lambda]}$.

A matrix representation for σ_0, π_0 is provided by (1.8), which in the case of ^{9}Li is

$$\sigma_0 = \mathbf{E}_6 \otimes \gamma_3 + \gamma_6 \otimes E_3, \qquad \pi_0 = E_6 \otimes \gamma_3 - \gamma_6 \otimes E_3 \tag{3.6}$$

where E is the unit matrix and γ_3, γ_6 are the Lie operators (2.3a) with μ replaced by $2\gamma_k$, $k = 3, 6$. With this representation, (3.5a) becomes

$$C_{[3303]} = \frac{2}{3}\{5E_6 \otimes \gamma_3^2 - 3\gamma_6^2 \otimes \gamma_3 + 17E_6 \otimes \gamma_3\} \tag{3.7}$$

The computed eigenvalues of (3.7) divided by 16 are the same as those of Table I, which is strong confirmation of the state labeling. The matrix is reducible and the submatrix containing the state $[\Lambda]$ with the eigenvalue $-3.5i$ is

$$\frac{A}{16} = \begin{array}{c} \left[\begin{array}{cccccccc}
-\frac{5}{4} & \frac{3}{4}\sqrt{5} & 0 & -\frac{\sqrt{15}}{2} & 0 & & & \\
\frac{3}{4}\sqrt{5} & -\frac{5}{4} & 0 & -2\sqrt{3} & \frac{3}{2}\sqrt{3} & & & \\
0 & 0 & -\frac{5}{4} & \frac{3}{4}\sqrt{5} & 0 & & & \\
-\frac{\sqrt{15}}{2} & -2\sqrt{3} & \frac{3}{4}\sqrt{5} & \boxed{\frac{11}{4}} & -3 & 0 & \frac{3}{2}\sqrt{3} & 0 \\
0 & \frac{3}{2}\sqrt{3} & 0 & \boxed{-3} & \frac{11}{4} & \frac{3}{4}\sqrt{5} & -2\sqrt{3} & -\frac{\sqrt{15}}{2} \\
 & & & 0 & \frac{3}{4}\sqrt{5} & -\frac{5}{4} & 0 & 0 \\
 & & & \frac{3}{2}\sqrt{3} & -2\sqrt{3} & 0 & -\frac{5}{4} & \frac{3}{4}\sqrt{5} \\
 & & & 0 & -\frac{\sqrt{15}}{2} & 0 & \frac{3}{4}\sqrt{5} & -\frac{5}{4}
\end{array}\right] \end{array}
\begin{array}{c} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \\ X_7 \\ X_8 \end{array} \tag{3.8a}$$

with eigenvalues

$$\{-5;\ -3.5;\ -2.5;\ -2;\ -1;\ -0.5;\ 2.5;\ 10\} \tag{3.8b}$$

corresponding to the states marked with an asterisk in Table I. The remaining eight states will be opposite in sign, but will include [3330] and [3312]. Unfortunately, it is not possible to assign these states uniquely to the rows of (3.8a), which have simply been labeled by the coordinates X_i ($i = 1, \ldots, 8$). Now we add 5 to the set (3.8b) and divide by 3 to get the canonical set

$$\left\{0; \frac{1}{2}, \frac{5}{6}; 1; \frac{4}{3}, \frac{3}{2}, \frac{5}{2}; 5\right\} \tag{3.8c}$$

so that the complex structure of the Kähler manifold is given

$$\mu = \frac{1}{48}\begin{bmatrix} & A \\ -A & \end{bmatrix} + \frac{5}{3}\begin{bmatrix} & E_8 \\ -E_8 & \end{bmatrix} = B + \frac{5}{3}\sigma \otimes E_8; \qquad \sigma = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \tag{3.9}$$

with the characteristic equation

$$F(\mu) = \mu\left(\mu^2 + \frac{1}{4}\right)\left(\mu^2 + \frac{25}{16}\right)(\mu^2 + 1)\left(\mu^2 + \frac{16}{9}\right) \times \left(\mu^2 + \frac{9}{4}\right)\left(\mu^2 + \frac{25}{4}\right)(\mu^2 + 25) \tag{3.10}$$

Then writing

$$e^{B\theta} = e^{\mu\theta}(e^{-(5/3)\sigma\theta} \otimes E_8); \qquad e^{-(5/3)\sigma\theta} = \begin{bmatrix} \cos\frac{5}{3}\theta & -\sin\frac{5}{3}\theta \\ \sin\frac{5}{3}\theta & \cos\frac{5}{3}\theta \end{bmatrix} \tag{3.11}$$

we can use (2.7) to calculate the complementary functions

$$64X_4 = \left(9\sin\frac{\theta}{2} + 5\sin\frac{5}{6}\theta + \frac{3}{2}\sin\theta + \frac{1}{2}\sin\frac{4}{3}\theta + 3\sin\frac{3}{2}\theta + 15\sin\frac{5}{2}\theta + 22.5\sin 5\theta\right)\sin\frac{5}{3}\theta \tag{3.12a}$$

$$64X_5 = \left(9\sin\frac{\theta}{2} + 5\sin\frac{5}{6}\theta - \frac{3}{2}\sin\theta - \frac{1}{2}\sin\frac{4}{3}\theta + 3\sin\frac{3}{2}\theta + 15\sin\frac{5}{2}\theta - 22.5\sin 5\theta\right)\sin\frac{5}{3}\theta = -X_4(1080^\circ - \theta) \tag{3.12b}$$

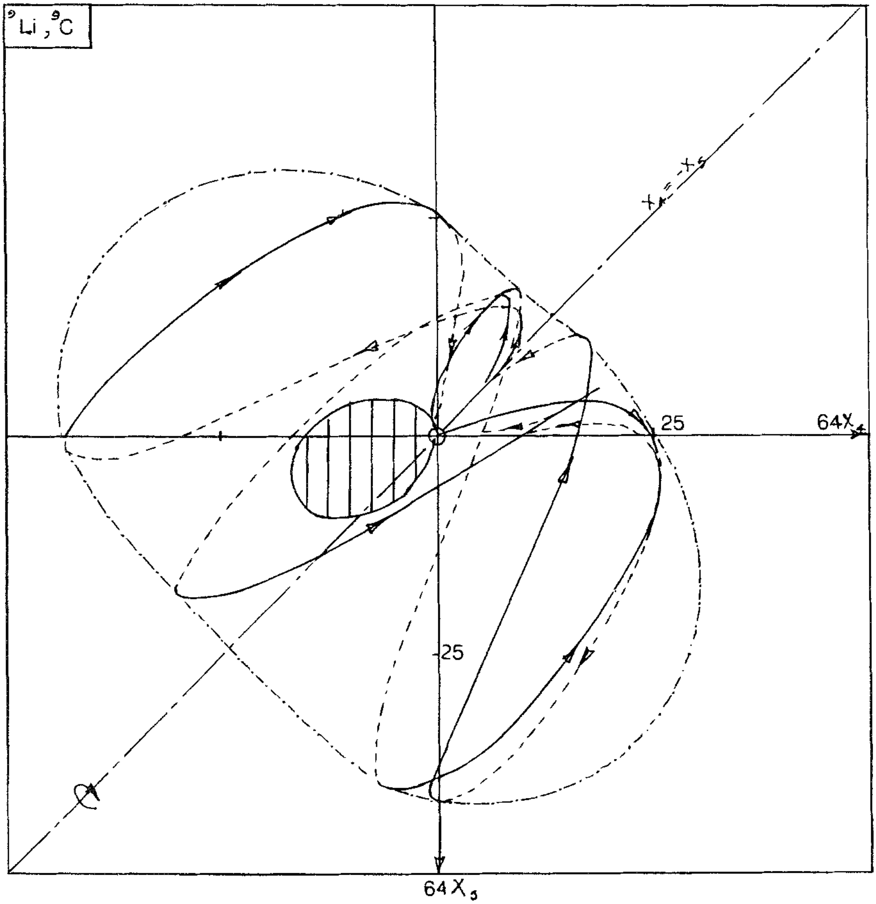

Fig. 1. Geodesics on the manifold of ^{9}Li, ^{9}C.

which are in the box in the fourth column of (3.8a) (or the 12th column of B) and assumed to be associated with $[\Lambda]$. We may now use a theorem of Kobayashi and Nomizu (1969, Chapter X, §2) to plot Fig. 1, which is Fig. 2 of de Wet (1994) redrawn to show the cross section of a geodesic starting from the origin and lying on a distorted torus rotating about $X_4, -X_5$. Other geodesics on this Kähler manifold can be obtained by rotations which fix the origin. To show that this geodesic field corresponds to the motion of nucleons, we calculate the charge density $\rho = e\psi\psi^\dagger$, where ψ is the entire wave function of (2.7), namely that corresponding to the whole of the 12th row of B which will also include the cosine terms. We find the term at the intersection of the 12th row and column is

$$\psi\psi^{\dagger} = [e^{2\mu\theta}]_{12,12} \cos^2 \frac{5}{3}\,\theta$$
$$= \frac{1}{64}\left(\frac{15}{2} + 9\cos\theta + 5\cos\frac{5}{8}\,\theta + \cdots + \frac{45}{2}\cos 10\,\theta\right)\cos^2\frac{5}{3}\,\theta$$

so integrating over a strip $64/15\pi$ units wide (87 units to the scale of Fig. 1) and along a length $12\pi = 2160°$, corresponding to one complete cycle, we confirm that

$$\rho = (N - Z)e = 3e$$

This is equivalent to a 20-fold figure 8 winding around the torus.

Returning to (3.12), multiple-angle formulas may be used to express sin 5θ in terms of sin θ; {sin $5/2\theta$; sin $3/2\theta$} in terms of sin $\theta/2$; and sin $8/6\ \theta$ in terms of sin $5/6\ \theta$; so that there are actually three different sets of translated chiral-spin angular momenta. Chiral-spin has been associated with fractional statistics by Wen *et al.* (1989) and it has been emphasized by Lerda (1992) that any spin, or fractional statistic, is possible for nucleons moving on a surface. These are called anyons and further show that the Kähler manifold carries a nuclear field.

The eigenvalues (3.8c) are somewhat arbitrary because the theorem only requires that one member of the set be unity. However, by (2.7) the coefficients $F_k(\mu)/F_k(i\lambda_k)$ will not be altered by another choice $\lambda_f \to \lambda_f/n$ (or $\mu \to \mu/n$). There will, however, also be a change of the argument $\lambda\theta$ to $\lambda\ \theta/n$, but θ/n can simply be represented by another parameter φ, so the shape of the geodesics will be unaffected.

Using (3.11), it follows that (2.12) is now

$$T = -\bar{T}^t = \mu \tan \lambda_t\theta \sum_{k=1,\ldots}^{n} \frac{i(F_k(\mu)/\mu)}{F_k(i\lambda_k)} \tan \lambda_k\theta \tag{3.13}$$

where $\lambda_t = 5/3$.

Thus $dT|_{T=0} = 0$, so the metric (2.10a) for this case is quite different and no longer flat. We have a distorted torus with

$$dT = \mu\, d\theta \left\{ \tan \lambda_t\theta \sum_k K_k(\mu) \sec^2\lambda_k\theta \right\}$$
$$T\bar{T}^t = \tan^2\lambda_t\theta \sum_k K_k(\mu) \tan^2\lambda_k\theta$$
$$(1 + T\bar{T}^t)^{-1} = 1 + \sum_k K_k(\mu)\{(1 + \tan^2\lambda_t\theta \tan^2\lambda_k\theta)^{-1} - 1\} \tag{3.14}$$

and

$$\frac{dT}{1 + T\bar{T}^t} = \mu \, d\theta \sum_k K_k(\mu) \frac{\tan \lambda_t\theta \sec^2\lambda_k\theta}{1 + \tan^2\lambda_t\theta \tan^2\lambda_k\theta}$$
$$= d\bar{T}^t(1 + T\bar{T}^t)^{-1} \qquad \text{if} \quad \mu \to \bar{\mu} \tag{3.15}$$

From (3.15) we can easily find the nonzero components

$$\Gamma^a_{bc} = g^{a\bar{d}} \frac{\partial}{\partial b} g_{c\bar{d}}, \qquad \Gamma^{\bar{a}}_{\bar{b}\bar{c}} = g^{\bar{a}d} \frac{\partial}{\partial \bar{b}} g_{\bar{c}d} \tag{3.16}$$

of the affine connection with $b = \lambda_t\theta, \bar{b} = -\lambda_t\theta$ (Green *et al.*, 1988, §15.3.3). But in view of (2.10) we will only be interested in

$$\Gamma^{\bar{k}}_{\bar{b}\bar{k}} = g^{\bar{k}k} \frac{\partial}{\partial \bar{b}} g_{\bar{k}k} = \frac{\partial}{\partial \bar{b}} \ln \det g \tag{3.17}$$

where $k = \lambda_k\theta$ and $\bar{k} = -\lambda_k\theta$. From (3.17) we deduce that the components

$$R_{\bar{b}c} = -\frac{\partial}{\partial c} \Gamma_{\bar{b}}{}^{\bar{k}}{}_{\bar{k}}$$

of the Ricci curvature tensor will vanish, which is a requirement of those Kähler manifolds referred to as Calabi–Yau spaces by string theorists. However, using the definition given by Kobayashi and Nomizu (1969, Chapter IX, §7), the sectional curvature $R_{k\bar{k}k\bar{k}}$ of each plane p, with orthogonal basis $k, \bar{k}$, will not vanish. We find instead that

$$R_{k\bar{k}k\bar{k}} = \frac{\partial^2}{\partial k \partial \bar{k}} g_{k\bar{k}} - g^{k\bar{k}} \frac{\partial g_{k\bar{k}}}{\partial k} \frac{\partial g_{k\bar{k}}}{\partial \bar{k}} = \frac{\partial^2 g_{k\bar{k}}}{\partial k \partial \bar{k}} - \sum \frac{\partial^2 g_{j\bar{j}}}{\partial j \partial \bar{j}} \tag{3.18}$$

so curvature is determined by the orientation of the remaining p-planes. This means that a spinor field corresponding to the state $[\lambda_k]$ and propagated parallelly only around the section $k\,\bar{k}$ will return to its original value. But this is precisely the condition found by Green *et al.* (1988, Chapter 15) to show that a Calabi–Yau space carries a string field. Therefore chiral spinor fields define the same space for nucleons and strings!

4. CONCLUSION

It should be emphasized that half the states of Table I will not exhibit a complex structure because $(\lambda_2 + \lambda_3)$ is even. Therefore $C_{[\lambda]}$ will be a polynomial in even values of σ_0, π_0 and belong to the vertical subspace h. The characteristic equation will now take the form (2.1a), and (2.1b) will yield a representation of the orthogonal group $O(p)$. An example is ^{8}Li, discussed by de Wet (1994), but there Fig. 1 is misleading because a geodesic field does not apply to representations on h. However, representations of $O(p)$ will motivate a shell structure except that the orbits need not define a two-dimensional surface.

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